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FORCED VIBRATION ANALYSIS OF ISOTROPIC THIN RECTANGULAR PLATES

Njoku F. C.[1], Onyechere I. C.[2*], Onwuka D. O.[3], Ibearugbulem O. M.[4], Igbojiaku A. U.[5]
[1,2,3,4,5.] Department of Civil Engineering, Federal University of Technology, Owerri, Nigeria.
[*]Corresponding Author's E-mail: onyecherechigozie@gmail.com; onyechere.chigozie@futo.edu.ng

ABSTRACT

This work used the general shape function assumed by Szilard (2004) to formulate the solution to the forced vibration equation of an isotropic thin rectangular plate. By applying the appropriate boundary conditions on dimensionless co-ordinates (ζ, η) it produced the shape function of a rectangular plate with opposite edges clamped, one edge of the other opposite edges clamped, and the other edge is simply supported, this is denoted as 'CCSC' plate, in terms of a deflection constant, A. It converted the forced vibration equation to an energy equation by multiplying it with a deflection term, w, and integrating over the whole surface of the plate. By substituting the shape function and aspect ratio of the plate into the energy equation, the value of A was obtained, from which the full deflection equation was derived. Then the shear force, deflections, bending and twisting moments were obtained from the deflection equation. The values of deflections and bending moments obtained satisfy the natural and geometric boundary conditions of the plate.

Keywords: Isotropic, Thin Plates, Free and Forced Vibration, Fundamental Frequency, Frequency Factor, Dynamic load factor.

1. INTRODUCTION

Forced vibration analysis of isotropic thin rectangular plates has been of great interest in the field of structural mechanics and dynamics because of it is applied in several engineering fields. A comprehensive knowledge of the behavior of such plates when carrying external forces is essential in enabling engineers to optimize the design of structures to guarantee their reliability and performance (Adamou, 2024). Among the various boundary conditions of rectangular plates, of peculiar interest is the CCSC plate, where opposite edges of the plate are clamped, one edge of the other opposite edges is clamped while the other edge is simply supported (Aginam et al., 2021).

The bodies of vehicles, ships and aircrafts are made of plates. They are subjected to vibration arising from the engines, rough roads and airstrips. The floors and walls of structures are made of plates. They suffer vibration from earthquakes, earth-moving equipment, bombs and live loads from human occupants, machinery and furniture (Neya & Nateghi-Babagi, 2025). The CCSC plate boundary condition has a unique geometry that presents complex vibrational behavior when acted upon by external forces. The clamped/fixed edges prevents the movement of the plate, leading to specific mode shapes and frequencies that are different from that of plates with other boundary conditions (Narita, 2023). In carrying out forced vibration analysis of CCSC plates careful considerations of these boundary conditions are required to be taken into account so as to correctly determine the behavior of the plate under the influence of external excitations . Ventsel & Krauthammer, (2001)considered two kinds of vibration of plates, namely, free and forced vibrations. Free or natural vibration occurs in the absence of applied loads, but may be triggered off by applying initial conditions to the plate (Onyechere et al., 2021). Free vibration analysis produces the natural or fundamental frequency of the plate which depends on the geometry and material properties of the plate (Jadee et al., 2020). Forced vibration results from an application of time-dependent loads, which are called dynamic loads. Forced vibration analysis produces the deflection equation, bending moments and shear forces resulting from dynamic loads(Onyechere et al., 2020).

Many researchers have used various methods to determine the fundamental frequency of a CCSC plate. Papkov & Banerjee, (2023) used superposition approach by applying modified trigonometric expressions as shear deformation functions and serries expressions displacement functions in the analysis isotropic and orthotropic plates. Adamou, (2024) applied polynomial functions in the Ritz approach in the free vibration analysis of thin rectangular plates. Wang et al., (2025) applied Tailor series functions and differential quadrature method in studying the in-plane vibrational behaviour of rectangular plates resting on elastic boundaries. . (Szilard, 2004)presented a forced vibration analysis of a CCSC plate by assuming a forcing function, but he did not relate it to the fundamental natural frequency of the plate. Thus, much work has not been done in the area of forced vibration.

The present study therefore, used the general shape function obtained by (Szilard, 2004) to determine the exact deflection equations for a CCSC plate under the influence of forced vibrations and to obtain the shear forces, bending and twisting moments on a CCSC plate under the influence of forced vibrations.

2. METHODOLOGY

2.1 Exact Solution to the Governing Differential Equation

(Ventsel & Krauthammer, 2001)obtained the governing differential equation of motion of thin plate under *forced*, *undamped* motion as follows:

$$D\left(\frac{\partial^4}{\partial x^4}+\frac{2\,\partial^4}{\partial x^2\,\partial y^2}+\frac{\partial^4}{\partial y^4}\right)W_{(x,y,t)} = q_{(x,y,t)} - m\left(\frac{\partial^2 w}{\partial t^2}\right)_{(x,y,t)} \qquad (1)$$

Where: q represents plate loading per unit area; $w_{(x,y,t)}$ represent displacement function; m = ϱh which represents mass of plate per unit area; ϱ is the mass density of plate material; h is the plate thickness; t = time; x, y = co-ordinates; a, b = plate dimensions in x and y axes respectively.

$$D = \frac{Eh^3}{12(1-v^2)} \qquad (2)$$

Where: D represents flexural rigidity of the plate material; E represents modulus of elasticity; v represents Poisson's ratio

From the works of (Szilard, 2004), for a harmonic vibration, the load and deflection functions can be assumed to be as given in equations (3) and (4) respectively.

$$q_{(x,y,t)} = q_{(x,y,)}\sin\omega t \qquad (3)$$

$$w_{(x,y,t)} = w_{(x,y,)}\sin\omega t \qquad (4)$$

where ω is the fundamental natural frequency of the plate, and $w_{(x,\,y)}$ is the shape function of the plate.

For convenience let us express $w_{(x,\,y)}$ in terms of dimensionless co-ordinates, ζ and η, as shown in Figure 1), that is:

$$\zeta = \frac{x}{a} \qquad (5)$$

$$\eta = \frac{y}{b} \qquad (6)$$

where: a and b = plate dimensions, as shown in Figure 1.

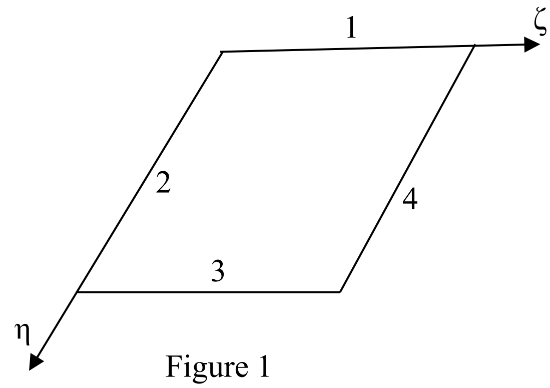

Figure 1

From (Chakraverty, 2009; Ibearugbulem et al., 2014; Szilard, 2004) the exact solution can be assumed to be a polynomial in x and y (or, ζ and η, for the dimensionless co-ordinate system). So, we have:

$$w_{(x,y)} = w_{(\zeta,\eta)} = AS_p \qquad (7)$$

where: A represents Deflection constant, S_p represents a polynomial function in terms of ζ and η coordinates

By substituting Equation (7) into Equation (4), the exact solution of Equation (1) becomes Equation (8).

$$w_{(x,y,t)} = w_{(\zeta,\eta,t)} = AS_p\sin\omega t \qquad (8)$$

2.2 Shape Function of CCSC Plate

By applying the boundary conditions to the deflection functions of a two-dimensional plate, (Ibearugbulem et al., 2014)obtained Equation (9) as the shape function of a CCSC plate.

$$w = A(\zeta^2 - 2\zeta^3 + \zeta^4)(1.5\eta^2 - 2.5\eta^3 + \eta^4) \qquad (9)$$

Comparing Equation (9) with Equation (7), the shape function is obtained in non-dimensional form as;

$$S_p = (\zeta^2 - 2\zeta^3 + \zeta^4)(1.5\eta^2 - 2.5\eta^3 + \eta^4) \qquad (10)$$

2.3 Fundamental Natural Frequencies of CCSC Plate

(Ibearugbulem et al., 2014)obtained Equation (11) as the fundamental natural frequency $'\omega'$ of a thin rectangular isotropic plate.

$$\omega = \frac{H_{b\beta}}{b^2}\sqrt{\frac{D}{m}} \qquad (11)$$

where $H_{b\beta}$ is a numerical coefficient called the non-dimensional frequency parameter, expressed in terms of the dimension 'b', for a plate of aspect ratio $\beta = \frac{a}{b}$.

$$H_{b\beta} = \sqrt{\left(\frac{C_2}{B_2}\right)} \qquad (12)$$

Where;

$$B_2 = \int_0^1\int_0^1 {S_p}^2\,\partial\zeta\,\partial\eta \qquad (13)$$

$$C_2 = \int_0^1\int_0^1 (K_2 S_p)\,\partial\zeta\,\partial\eta \qquad (14)$$

$$K_2 = \left[\frac{1}{\beta^4}\left(\frac{\partial^4 S_p}{\partial\zeta^4}\right) + \frac{2}{\beta^2}\left(\frac{\partial^4 S_p}{\partial\zeta^2\,\partial\eta^2}\right) + \left(\frac{\partial^4 S_p}{\partial\eta^4}\right)\right] \qquad (15)$$

The aspect ratio $'\beta'$ is given as;

$$\beta = \frac{a}{b} \qquad (16)$$

For a CCSC plate $H_{b\beta}$ was evaluated as:

$$H_{b\beta} = \sqrt{\left(\frac{503.9894719}{\beta^4} + \frac{272.8363971}{\beta^2} + 238.7318683\right)} \qquad (17)$$

2.4 Determination of the Deflection Constant (A)

Before collapse can occur, the external work done by the dynamic load, q, must be equal to the internal strain energy of the plate. This satisfies the principle of conservation of energy.

Substituting Equations (3) and (4) into Equation (1) we get Equation (18).

$$D\sin\omega t\left(\frac{\partial^4}{\partial x^4} + \frac{2\,\partial^4}{\partial x^2\,\partial y^2} + \frac{\partial^4}{\partial y^4}\right)W_{(x,y)} = q_{(x,y)}\sin\omega t - m\omega^2 w\sin\omega t \qquad (18)$$

$$\frac{\partial^4 w}{\partial x^4} + \frac{2\,\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4} - \frac{q}{D} - \frac{m\omega^2 w}{D} = 0 \qquad (19)$$

By dividing Equation (18) with DSinωt we obtain Equation 19). where w and q represent $w_{(x,\,y)}$ and $q_{(x,\,y)}$ respectively.

Rearranging yields:

$$q = D\left(\frac{\partial^4 w}{\partial x^4} + \frac{2\,\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) - m\omega^2 w \qquad (20)$$

In this form, both sides of the equation represent force per unit area. The left-hand side represents external forces, while the right-hand side represents internal forces.

Multiplying both sides of Equation (3.20) by $\frac{1}{2}$w, gives:

$$\frac{1}{2}qw = \frac{1}{2}D\left(\frac{\partial^4 w}{\partial x^4} + \frac{2\,\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4}\right)w - \frac{1}{2}m\omega^2 w^2 \qquad (21)$$

The left-hand side represents external work done while the right-hand side represents internal strain energy, all per unit area of the plate.

Integrating both sides of the Equation (21) yields the total external work and the total internal strain energy for the whole plate. Thus, integrating the equation with respect to x and y:

$$\int_0^b\int_0^a (\left(\frac{\partial^4 w}{\partial x^4}\right)w + \left(\frac{2\,\partial^4 w}{\partial x^2\,\partial y^2}\right)w + \left(\frac{\partial^4 w}{\partial y^4}\right)w - \left(\frac{qW}{D} - \frac{m\omega^2 w^2}{D}\right))\,\partial x\,\partial y = 0 \qquad (22a)$$

$$\int_0^b\int_0^a \left(\left(\frac{\partial^4 w}{\partial x^4}\right)w + \left(\frac{2\,\partial^4 w}{\partial x^2\,\partial y^2}\right)w + \left(\frac{\partial^4 w}{\partial y^4}\right)w - \frac{qW}{D} - \frac{m\omega^2 w^2}{D}\right)\partial x\,\partial y = 0 \qquad (22b)$$

The Equation (22b) is called the Energy Equation.

After substituting the expressions ζ, η and β (from Equations (5), (6) and (6)), Equation (22) becomes:

$$\int_0^b\int_0^a \left(\left(\frac{\partial^4 w}{a^4\,\partial\zeta^4}\right)w + \frac{2}{a^2 b^2}\left(\frac{\partial^4 w}{\partial\zeta^2\,\partial\eta^2}\right)w + \frac{1}{b^4}\left(\frac{\partial^4 w}{\partial\eta^4}\right)w - \frac{qW}{D} - \frac{m\omega^2 w^2}{D}\right)ab\,\partial\zeta\,\partial\eta = 0 \qquad (23)$$

Where; $\partial x = a\partial\zeta$ and $\partial y = b\partial\eta$.

Multiplying Equation (23) by b^4, Equation (24) is obtained.

$$ab\int_0^b\int_0^a \left(\frac{b^4}{a^4}\left(\frac{\partial^4 w}{\partial\zeta^4}\right)w + \frac{2b^2}{a^2}\left(\frac{\partial^4 w}{\partial\zeta^2\,\partial\eta^2}\right)w + \left(\frac{\partial^4 w}{\partial\eta^4}\right)w - \frac{b^4 qW}{D} - \frac{b^4 m\omega^2 w^2}{D}\right)\partial\zeta\,\partial\eta = 0 \qquad (24)$$

Putting $\beta = \frac{a}{b}$ and substituting in Equation (24), Equation (25) is obtained.

$$\int_0^1\int_0^1\left(\frac{1}{\beta^4}\left(\frac{\partial^4 w}{\partial\zeta^4}\right)w+\frac{2}{\beta^2}\left(\frac{\partial^4 w}{\partial\zeta^2\,\partial\eta^2}\right)w+\left(\frac{\partial^4 w}{\partial\eta^4}\right)w-\frac{b^4 qW}{D}-\frac{b^4 m\omega^2 w^2}{D}\right)\partial\zeta\,\partial\eta=0 \quad (25)$$

$$\int_0^1\int_0^1\left(\frac{1}{\beta^4}\left(\frac{\partial^4 w}{\partial\zeta^4}\right)w+\frac{2}{\beta^2}\left(\frac{\partial^4 w}{\partial\zeta^2\,\partial\eta^2}\right)w+\left(\frac{\partial^4 w}{\partial\eta^4}\right)w\right)\partial\zeta\,\partial\eta-\int_0^1\int_0^1\left(\frac{b^4 qW}{D}\right)\partial\zeta\,\partial\eta-\int_0^1\int_0^1\left(\frac{b^4 m\omega^2 w^2}{D}\right)\partial\zeta\,\partial\eta=0 \quad (26)$$

For free vibration, the load in equation (26) is set to zero (i.e. q = 0), so the Equation (26) becomes:

$$\int_0^1\int_0^1\left(\frac{1}{\beta^4}\left(\frac{\partial^4 w}{\partial\zeta^4}\right)w+\frac{2}{\beta^2}\left(\frac{\partial^4 w}{\partial\zeta^2\,\partial\eta^2}\right)w+\left(\frac{\partial^4 w}{\partial\eta^4}\right)w\right)\partial\zeta\,\partial\eta-\int_0^1\int_0^1\left(\frac{b^4 m\omega^2 w^2}{D}\right)\partial\zeta\,\partial\eta=0 \quad (27)$$

Substituting equation (7) into the free vibration equation (27) yields equation (28).

$$\int_0^1\int_0^1\left(\frac{1}{\beta^4}\left(\frac{\partial^4 AS_p}{\partial\zeta^4}\right)AS_p+\frac{2}{\beta^2}\left(\frac{\partial^4 AS_p}{\partial\zeta^2\,\partial\eta^2}\right)AS_p+\left(\frac{\partial^4 AS_p}{\partial\eta^4}\right)AS_p\right)\partial\zeta\,\partial\eta-\int_0^1\int_0^1\left(\frac{b^4 m\omega^2 (AS_p)^2}{D}\right)\partial\zeta\,\partial\eta=0 \quad (28)$$

Since A is a constant, equation (28) can be re-written as:

$$A^2\int_0^1\int_0^1\left(\frac{1}{\beta^4}\left(\frac{\partial^4 S_p}{\partial\zeta^4}\right)S_p+\frac{2}{\beta^2}\left(\frac{\partial^4 S_p}{\partial\zeta^2\,\partial\eta^2}\right)S_p+\left(\frac{\partial^4 S_p}{\partial\eta^4}\right)S_p\right)\partial\zeta\,\partial\eta-A^2\int_0^1\int_0^1\left(\frac{b^4 m\omega^2 S_p^{\,2}}{D}\right)\partial\zeta\,\partial\eta=0 \quad (29)$$

$$\text{Let } K_2=\frac{1}{\beta^4}\left(\frac{\partial^4 S_p}{\partial\zeta^4}\right)+\frac{2}{\beta^2}\left(\frac{\partial^4 S_p}{\partial\zeta^2\,\partial\eta^2}\right)+\left(\frac{\partial^4 S_p}{\partial\eta^4}\right) \quad (30)$$

Substituting the equation (30) into the equation (29) gives equation (31).

$$\int_0^1\int_0^1 (K_2\,S_p)\,\partial\zeta\,\partial\eta=\frac{b^4 m\omega^2}{D}\int_0^1\int_0^1 S_p^{\,2}\,\partial\zeta\,\partial\eta \quad (31)$$

$$Let\ B_2=\int_0^1\int_0^1 S_p^{\,2}\,\partial\zeta\,\partial\eta \quad (32)$$

and $C_2=\int_0^1\int_0^1 (K_2\,S_p)\,\partial\zeta\,\partial\eta$ (33a)

Substituting the equations (32) and (33a) into the equation (31), we get:

$$C_2=\frac{b^4 m\omega^2 B_2}{D} \quad (33b)$$

and rearranging, yields equation (34). $\omega 2=\frac{C_2}{b^4 m}\frac{D}{B_2}$ (34)

For forced vibration, ω is replaced with Ω, the forcing frequency on the plate, so the equation (26) becomes:

$$\int_0^1\int_0^1\left(\frac{1}{\beta^4}\left(\frac{\partial^4 w}{\partial\zeta^4}\right)w+\frac{2}{\beta^2}\left(\frac{\partial^4 w}{\partial\zeta^2\,\partial\eta^2}\right)w+\left(\frac{\partial^4 w}{\partial\eta^4}\right)w\right)\partial\zeta\,\partial\eta-\int_0^1\int_0^1\left(\frac{b^4 qW}{D}\right)\partial\zeta\,\partial\eta-\int_0^1\int_0^1\left(\frac{b^4 m\Omega^2 w^2}{D}\right)\partial\zeta\,\partial\eta=0 \quad (35)$$

Substituting the Equation .(7) into the forced vibration Equation (35), gives Equation (36a).

$$\int_0^1\int_0^1\left(\frac{1}{\beta^4}\left(\frac{\partial^4 w}{\partial\zeta^4}\right)w+\frac{2}{\beta^2}\left(\frac{\partial^4 w}{\partial\zeta^2\,\partial\eta^2}\right)w+\left(\frac{\partial^4 w}{\partial\eta^4}\right)w\right)\partial\zeta\,\partial\eta-\int_0^1\int_0^1\left(\frac{b^4 qW}{D}\right)\partial\zeta\,\partial\eta-\int_0^1\int_0^1\left(\frac{b^4 m\Omega^2 w^2}{D}\right)\partial\zeta\,\partial\eta=0 \quad (36a)$$

$$\int_0^1\int_0^1\left(\frac{1}{\beta^4}\left(\frac{\partial^4 (AS_p)}{\partial\zeta^4}\right)AS_p+\frac{2}{\beta^2}\left(\frac{\partial^4 AS_p}{\partial\zeta^2\,\partial\eta^2}\right)AS_p+\left(\frac{\partial^4 AS_p}{\partial\eta^4}\right)AS_p\right)\partial\zeta\,\partial\eta-\int_0^1\int_0^1\left(\frac{b^4 qAS_p}{D}\right)\partial\zeta\,\partial\eta-\int_0^1\int_0^1\left(\frac{b^4 m\Omega^2 (AS_p)^2}{D}\right)\partial\zeta\,\partial\eta=0 \quad (36b)$$

The Equation (36b) can be re-written as follows:

$$A^2\int_0^1\int_0^1\left(\frac{1}{\beta^4}\left(\frac{\partial^4 S_p}{\partial\zeta^4}\right)S_p+\frac{2}{\beta^2}\left(\frac{\partial^4 S_p}{\partial\zeta^2\,\partial\eta^2}\right)S_p+\left(\frac{\partial^4 S_p}{\partial\eta^4}\right)S_p\right)\partial\zeta\,\partial\eta-A^2\int_0^1\int_0^1\left(\frac{b^4 m\Omega^2 {S_p}^2}{D}\right)\partial\zeta\,\partial\eta=A\int_0^1\int_0^1\left(\frac{b^4 q S_p}{D}\right)\partial\zeta\,\partial\eta \quad (37)$$

By putting the Equation (30) into the Equation (37), Equation (38) is obtained.

$$A^2\int_0^1\int_0^1 (K_2\, S_p)\,\partial\zeta\,\partial\eta-A^2\frac{b^4 m\Omega^2}{D}\int_0^1\int_0^1 {S_p}^2\,\partial\zeta\,\partial\eta=A\frac{b^4 q}{D}\int_0^1\int_0^1 S_p\,\partial\zeta\,\partial\eta \quad (38)$$

Putting the Equations (32) and (33) into the Equation (38 results into Equation (39).

$$A\left(C_2-\frac{b^4 m\Omega^2}{D}B_2\right)=\frac{b^4 q}{D}\int_0^1\int_0^1 S_p\,\partial\zeta\,\partial\eta \quad (39)$$

Rearranging Equation (39) gives:

$$A=\frac{\frac{b^4 q}{D}\int_0^1\int_0^1 S_p\ \partial\zeta\,\partial\eta}{\left(C_2-\frac{b^4 m\Omega^2}{D}B_2\right)} \quad (40)$$

The Equation (40) gives the deflection constant A of forced vibration, while $S_p$is the deflection polynomial in ζ and η cordinate.

2.5 Determination of the Dynamic Load Factor (DLF)

The dynamic load factor (DLF) relates the dynamic response of a point (x, y) on the plate to the static response of the same point under the same loading. It is the ratio of dynamic deflection to static deflection at a point in the plate.

Under static loading conditions (pure bending), there is no vibration (Ω = 0), so Equation (40) becomes:

$$A=\frac{\frac{b^4 q}{D}\int_0^1\int_0^1 S_p\ \partial\zeta\,\partial\eta}{C_2} \quad (41)$$

So, the dynamic load factor (DLF) can be obtained by dividing Equation (40) with Equation (41), thus:

$$\mathrm{DLF}=\frac{C_2}{\left(C_2-\frac{b^4 m\Omega^2}{D}B_2\right)} \quad (42)$$

After dividing numerator and denominator by C_2, Equation (42) becomes:

$$\mathrm{DLF}=\frac{1}{\left(1-\frac{b^4 m\Omega^2}{D}\frac{B_2}{C_2}\right)} \quad (43)$$

But, substituting $\omega^2=\frac{C_2}{b^4 m}\frac{D}{B_2}$ from Equation (34) in the denominator of Equation (43), we get:

$$\mathrm{DLF}=\frac{1}{\left(1-\frac{\Omega^2}{\omega^2}\right)} \quad (44)$$

Equation (44) can be re-written as:

$$\mathrm{DLF}=\frac{1}{(1-N^2)} \quad (45)$$

where N is called the frequency factor, representing the ratio of forcing frequency (for forced vibration) to fundamental natural frequency (for free vibration).

$$N=\frac{\Omega}{\omega},\quad (0\le N^2\le 1) \quad (46)$$

2.6 Determination of the Numerical Deflection Coefficient G

Rearranging Equation (34), we obtain Equation (47).

$$\frac{b^4 m B_2}{D}=\frac{C_2}{\omega^2} \quad (47)$$

By substituting Equation (47) into Equation (40) we get Equation (48).

$$A=\frac{\frac{b^4 q}{D}\int_0^1\int_0^1 S_p\,\partial\zeta\,\partial\eta}{\left(C_2-\frac{\Omega^2}{\omega^2}C_2\right)} \quad (48)$$

Substituting Equation (45) into Equation (48) and simplifying we get Equation (49).

$$A=\frac{\frac{b^4 q}{D}\int_0^1\int_0^1 S_p\,\partial\zeta\,\partial\eta}{C_2(1-N^2)} \quad (49)$$

Substituting Equation (45) into Equation (49) we get Equation (50).

$$A=\left(\frac{\frac{b^4 q}{D}\int_0^1\int_0^1 S_p\,\partial\zeta\,\partial\eta}{C_2}\right)(DLF) \quad (50)$$

Equation (49) can be re-written as:

$$\mathrm{A} = \frac{\mathrm{Gb^4 q}}{\mathrm{D}} \qquad (51)$$

Where G is a numerical deflection coefficient.

$$G = \frac{\int_0^1 \int_0^1 \mathrm{S_p}\ \partial\zeta\,\partial\eta}{\mathrm{C_2}(1 - \mathrm{N}^2)} \qquad (52)$$

$$G = \frac{\int_0^1 \int_0^1 \mathrm{S_p}\ \partial\zeta\,\partial\eta}{\mathrm{C_2}}(\mathrm{DLF}) \qquad (53)$$

2.7 Determination of the Dynamic Deflection Equation

From Equations (8) and (51), the deflection is given by Equation (54).

$$w_{(x,y,t)} = w_{(\zeta,\eta,t)} = \mathrm{Gb^4 q}S_p \frac{\sin \Omega t}{D} \qquad (54)$$

Putting Equation (46) into Equation (54) yields the final deflection equation as:

$$w_{(x,y,t)} = w_{(\zeta,\eta,t)} = \mathrm{Gb^4 q}S_p \frac{\sin \mathrm{N}\omega t}{D} \qquad (55)$$

2.8 Determination of the Slopes

The slopes in the ζ and η directions are given by the following derivatives: $\frac{\partial w_{(\zeta,\eta,t)}}{\partial\zeta}$ and $\frac{\partial w_{(\zeta,\eta,t)}}{\partial\eta}$, respectively.

2.9 Determination of Bending and Twisting Moments

The equations derived by (Szilard, 2004)for moments, M_x, M_y and M_{xy}, on the plate were used in this work. They are presented in Equations (56) to (58).

$$M_x = -D\left(\frac{\partial^2 \mathrm{w}}{\partial \mathrm{x}^2} + v\frac{\partial^2 \mathrm{w}}{\partial \mathrm{y}^2}\right) \qquad (56)$$

$$M_y = -D\left(\frac{\partial^2 \mathrm{w}}{\partial \mathrm{y}^2} + v\frac{\partial^2 \mathrm{w}}{\partial \mathrm{x}^2}\right) \qquad (57)$$

$$M_{xy} = M_{yx} = -D(1-v)\left(\frac{\partial^2 \mathrm{w}}{\partial \mathrm{x}\,\partial \mathrm{y}}\right) \qquad (58)$$

Expressing Equations (56), (57) and (58) in dimensionless form we get Equations (59), (60) and (61) for M_x, M_y and M_{xy}, respectively.

$$M_x = -D\left(\frac{\partial^2 \mathrm{w}}{\mathrm{a}^2\,\partial\zeta^2} + \frac{1}{b^2}\frac{\partial^2 \mathrm{w}}{\partial\eta^2}\right)$$

$$= -\frac{D}{\mathrm{a}^2}\left(\frac{\partial^2 \mathrm{w}}{\partial\zeta^2} + \mathrm{v}\beta^2\frac{\partial^2 \mathrm{w}}{\partial\eta^2}\right) \qquad (59)$$

$$M_y = -D\left(\frac{\mathrm{v}\,\partial^2 \mathrm{w}}{\mathrm{a}^2\,\partial\zeta^2} + \frac{1}{b^2}\frac{\partial^2 \mathrm{w}}{\partial\eta^2}\right)$$

$$= -\frac{D}{\mathrm{a}^2}\left(\frac{\mathrm{v}\,\partial^2 \mathrm{w}}{\partial\zeta^2} + \beta^2\frac{\partial^2 \mathrm{w}}{\partial\eta^2}\right) \qquad (60)$$

$$M_{xy} = M_{yx} = -\frac{\mathrm{D}\beta(1-\mathrm{v})}{\mathrm{a}^2}\frac{\partial^2 \mathrm{w}}{\partial\zeta\,\partial\eta} \qquad (61)$$

The bending moment, M_x, about x-axis is given by Equation (59). The twisting moment, M_{xy}, can be obtained from Equation (61).

2.10 Values of Bending and Twisting Moment Coefficients

For each value of aspect ratio $\beta = \frac{a}{b}$, the expressions for $w_{(x,y,t)}$, and the values of ζ and η, are substituted into the Equations (59) to (61) to obtain the bending and twisting moments at midspan and at the supports, in the form:

$$M_x = Zqb^2 \qquad (62)$$

$$M_y = Jqb^2 \qquad (63)$$

$$M_{yx} = Iqb^2 \qquad (64)$$

$$M_{xy} = Kqb^2 \qquad (65)$$

where: Z, J, I and K are numerical coefficients.

2.11 Determination of Shear Forces

The equations derived by (Szilard, 2004)for shears forces, Q_x and Q_y, on the plate were used in this work. They are presented in Equations (66) and (67).

$$Q_x = \frac{\partial \mathrm{M_x}}{\partial \mathrm{x}} + \frac{\partial \mathrm{M_{xy}}}{\partial \mathrm{y}} = -D\left(\frac{\partial^3 \mathrm{w}}{\partial \mathrm{x}^3} + \frac{\partial^3 \mathrm{w}}{\partial \mathrm{x}\,\partial \mathrm{y}^2}\right) \qquad (66)$$

$$Q_y = \frac{\partial \mathrm{M_y}}{\partial \mathrm{y}} + \frac{\partial \mathrm{M_{xy}}}{\partial \mathrm{x}} = -D\left(\frac{\partial^3 \mathrm{w}}{\partial \mathrm{x}^2\,\partial \mathrm{y}} + \frac{\partial^3 \mathrm{w}}{\partial \mathrm{y}^3}\right) \qquad (67)$$

The shear forces, Q_x and Q_y, for plates are obtained by expressing Equations (66) and (67) in a dimensionless form.

The shear force, Q_x, about x-axis is as follows:

$$Q_x = \frac{\partial \mathrm{M_x}}{\partial \mathrm{x}} + \frac{\partial \mathrm{M_{xy}}}{\partial \mathrm{y}}$$

$$= -\frac{\mathrm{D}}{\mathrm{a}^3}\left(\frac{\partial^3 \mathrm{w}}{\partial\zeta^3} + (2 - \mathrm{v})\beta^2\frac{\partial^3 \mathrm{w}}{\partial\zeta\,\partial\eta^2}\right) \qquad (68)$$

$$Q_y = \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} = -\frac{D}{a^3}\left(\beta^3 \frac{\partial^3 w}{\partial \eta^3} + (2 - v)\beta \frac{\partial^3 w}{\partial \eta \, \partial \zeta^2}\right) \quad (69)$$

The shear force, Q_y, about y-axis is given by Equation (69).

2.12 Values of Shear Forces Coefficients

For each value of the aspect ratio, β, the expressions for $w_{(x, y, t)}$ are substituted into Equations (68) and (69) to obtain, respectively, the shear forces (reactions) at the supports, in the form:

$$Q_x = Lqb \quad (70)$$

$$Q_y = Rqb \quad (71)$$

where: L and R are numerical coefficients.

3. Results and Discussion

3.1 Deflection Results

From Equation (55) the dynamic deflection equation is given by as Equation (72)

$$w_{(\zeta,\eta,t)} = Gb^4 q S_p \frac{\sin N\omega t}{D} \quad (72)$$

From Equation (52), G is obtained as in given in Equation (73).

$$G = \frac{0.0025}{(1 - N^2)(0.006031746(\frac{1}{\beta^4}) + 0.003265306(\frac{1}{\beta^2}) + 0.002857142857)} \quad (73)$$

So the final dynamic deflection is given by Equation (74).

$$w_{(\zeta,\eta,t)} = \frac{Gqb^4}{D}(\zeta^2 - 2\zeta^3 + \zeta^4)(1.5\eta^2 - 2.5\eta^3 + \eta^4)\mathrm{Sin}N\omega t \quad (74)$$

So, the slopes in the ζ and η directions are given by Equations (75) and (76) respectively.

$$\frac{\partial w_{(\zeta,\eta,t)}}{\partial \zeta} = \frac{Gqb^4}{D}(2\zeta - 6\zeta^2 + 4\zeta^3)(1.5\eta^2 - 2.5\eta^3 + \eta^4)\mathrm{Sin}N\omega t \quad (75)$$

$$\frac{\partial w_{(\zeta,\eta,t)}}{\partial \eta} = \frac{Gqb^4}{D}(\zeta^2 - 2\zeta^3 + \zeta^4)(3\eta - 7.5\eta^2 + 4\eta^3)\mathrm{Sin}N\omega t \quad (76)$$

The values of 'G' obtained from Equation (73) for different values of N, β and φ ($= \frac{b}{a} = \frac{1}{\beta}$) are presented in Table 1.

The sine function in the deflection equation (Equation (74)) obtained in this work helps to satisfy the natural boundary conditions because vibrational deflections are usually sinusoidal. Also, sine is the function that has a zero value when the time t = 0, at the onset of the vibration. So we can say that the amplitude of the vibration is given by Equation (75).

Table 1: Values of Numerical Coefficient of Deflection, G, for CCSC Plate

N / β	**0.000**	**0.200**	**0.400**	**0.600**	**0.800**	**1.000**	**N** / **Φ**
0.500	0.022237	0.023163	0.026472	0.034745	0.061769	∞	**2.0**

0.526	0.026751	0.027866	0.031846	0.041798	0.074308	∞	**1.9**
0.556	0.032664	0.034025	0.038886	0.051038	0.090734	∞	**1.8**
0.588	0.039834	0.041494	0.047422	0.062241	0.110651	∞	**1.7**
0.625	0.049265	0.051318	0.058649	0.076977	0.136847	∞	**1.6**
0.666	0.061159	0.063707	0.072809	0.095561	0.169887	∞	**1.5**
0.714	0.076992	0.080200	0.091657	0.120300	0.213866	∞	**1.4**
0.769	0.097554	0.101619	0.116136	0.152428	0.270984	∞	**1.3**
0.833	0.124437	0.129622	0.148140	0.194434	0.345660	∞	**1.2**
0.909	0.159810	0.166469	0.190250	0.249703	0.443917	∞	**1.1**
1.000	0.205690	0.214261	0.244869	0.321391	0.571362	∞	**1.0**

The amplitude 'A' is obtained from Equation (51) and presented here as Equation (77)

$$A = \frac{Gb^4q}{D} \qquad (77)$$

The amplitude occurs at the following values of time: $\frac{\pi}{2N\omega}, \frac{3\pi}{2N\omega}$. Since S_p is a polynomial in ζ and η, it means that the amplitude of vibration at any time varies with position (co-ordinates) on the plate.

3.2 Results of Bending and Twisting Moment Coefficients

The coefficients Z, J, I and K, obtained for bending and twisting moments, M_x, M_y, M_{yx} and M_{xy}, are presented in Table 2.

Table 2: Coefficients of Bending and Twisting Moments for CCSC Plate

S/N	Moment Type	Coefficients
1	M_x	$Z = -\frac{G((2-12\zeta+12\zeta^2)(1.5\eta^2-2.5\eta^3+\eta^4)+v\beta^2(\zeta^2-2\zeta^3+\zeta^4)(3-15\eta+12\eta^2))SinN\omega t}{\beta^2}$
2	M_y	$J = -\frac{G(v(2-12\zeta+12\zeta^2)(1.5\eta^2-2.5\eta^3+\eta^4)+\beta^2(\zeta^2-2\zeta^3+\zeta^4)(3-15\eta+12\eta^2))SinN\omega t}{\beta^2}$
3	M_{yx}	$I = -\frac{G(1-v)(2\zeta-6\zeta^2+4\zeta^3)(3\eta-7.5\eta^2+4\eta^3)SinN\omega t}{\beta}$
4	M_{xy}	$K = -\frac{G(1-v)(2\zeta-6\zeta^2+4\zeta^3)(3\eta-7.5\eta^2+4\eta^3)SinN\omega t}{\beta}$

3.3 Shear Force Coefficients

The coefficients L and R, obtained for shear forces, Q_x and Q_y, are presented in Table 3.

Table 3: Coefficients of Shear Force for CCSC Plate

S/N	Moment Type	Coefficients
1	Q_x	$L = -\frac{G((-12+24\zeta)(1.5\eta^2-2.5\eta^3+\eta^4)+(2-v)\beta^2(2\zeta-6\zeta^2+4\zeta^3)(3-15\eta+12\eta^2))SinN\omega t}{\beta^3}$
2	Q_y	$R = -\frac{G(\beta^3(\zeta^2-2\zeta^3+\zeta^4)(-15+24\eta)+\beta(2-v)(2-12\zeta+12\zeta^2)(3\eta-7.5\eta^2+4\eta^3))SinN\omega t}{\beta^3}$

The results of these checks show that the deflection and slopes obtained in this work satisfy the natural and geometric boundary conditions of the CCSC plate.

3.4 Checks on the Satisfaction of Boundary Conditions

Values of deflection and slope at the C edges and values of deflection and bending moment at the S edge of the plate must be zeros in order to satisfy the boundary conditions for a CCSC plate. At time t = 0, SinNωt = 0. So deflection for the plate will be zero. At other values of time, t, the deflection at the C boundaries must be zero.

So, from Equation (4.2):

At edge 1, $\zeta = \frac{1}{2}$ and $\eta = 0$. So, $w_{(\zeta,\eta,t)} = \frac{Gqb^4}{D}\left(\frac{1}{4} - \frac{1}{4} + \frac{1}{16}\right)(0 - 0 + 0)\mathrm{SinN\omega t} = 0$.

At edge 2, $\zeta = 0$ and $\eta = \frac{1}{2}$. So, $w_{(\zeta,\eta,t)} = \frac{Gqb^4}{D}(0 - 0 + 0)\left(\frac{3}{8} - \frac{5}{16} + \frac{1}{16}\right)\mathrm{SinN\omega t} = 0$.

At edge 3, $\zeta = \frac{1}{2}$ and $\eta = 1$. So, $w_{(\zeta,\eta,t)} = \frac{Gqb^4}{D}\left(\frac{1}{4} - \frac{1}{4} + \frac{1}{16}\right)(1.5 - 2.5 + 1)\mathrm{SinN\omega t} = 0$.

At edge 4, $\zeta = 1$ and $\eta = \frac{1}{2}$. So, $w_{(\zeta,\eta,t)} = \frac{Gqb^4}{D}(1 - 2 + 1)\left(\frac{3}{8} - \frac{5}{16} + \frac{1}{16}\right)\mathrm{SinN\omega t} = 0$.

At time t = 0, SinNωt = 0. So slopes for the plate will be zero. At other values of time, t, the slopes at the C boundaries must be zero.

From Equations (4.3) and (4.4):

At edge 1, $\zeta = \frac{1}{2}$ and $\eta = 0$. So, $\frac{\partial w_{(\zeta,\eta,t)}}{\partial \eta} = \frac{Gqb^4}{D}\left(\frac{1}{4} - \frac{1}{4} + \frac{1}{16}\right)(0 - 0 + 0)\mathrm{SinN\omega t} = 0$.

At edge 2, $\zeta = 0$ and $\eta = \frac{1}{2}$. So, $\frac{\partial w_{(\zeta,\eta,t)}}{\partial \zeta} = \frac{Gqb^4}{D}(0 - 0 + 0)\left(\frac{3}{8} - \frac{5}{16} + \frac{1}{16}\right)\mathrm{SinN\omega t} = 0$.

At edge 4, $\zeta = 1$ and $\eta = \frac{1}{2}$. So, $\frac{\partial w_{(\zeta,\eta,t)}}{\partial \zeta} = \frac{Gqb^4}{D}(2 - 6 + 4)\left(\frac{3}{8} - \frac{5}{16} + \frac{1}{16}\right)\mathrm{SinN\omega t} = 0$.

At time t = 0, SinNωt = 0. So bending moments for all plates will be zero. At other values of time, t, the bending moments coefficients Z and J at all S boundaries must be zero.

From Table (4.2):

At edge 3, $\zeta = \frac{1}{2}$ and $\eta = 1$. So, $J = -\frac{G(v(2 - 6 + 3)(1.5 - 2.5 + 1) + \beta^2(\frac{1}{4} - \frac{1}{4} + \frac{1}{16})(3 - 15 + 12))\mathrm{SinN\omega t}}{\beta^2} = 0$.

4. CONCLUSIONS

From the results of this work, the following conclusions were drawn.

The exact dynamic deflection equation $\left(w_{(\zeta,\eta,t)}\right)$ for a CCSC plate subjected to forced vibrations developed in this work varies linearly with the plate's dimension 'b' and the applied load 'q' as shown in Equation (74).

The numerical values of the coefficient of deflection 'G' for CCSC plate subjected to forced vibration shown in Table 1 at each aspect ratio $\left(\beta = \frac{a}{b}\right)$ increases as the frequency factor 'N' increases. Also at the value of each frequency factor $\left(N = \frac{\Omega}{\omega}\right)$, the numerical values of the coefficient of deflection 'G' increases as the aspect ratio increases.

The bending and twisting moments of a CCSC plate subjected to forced vibration given equations (59) to (61) increase as the aspect ratio increase. The Shear force of a CCSC plate subjected to forced vibration given equations (68) and (69) increases as the aspect ratio increases but decreases as the plate's dimension 'a' increase.

The results of these checks show that the deflection and slopes obtained in this work satisfy the natural and geometric boundary conditions of the CCSC plate

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